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Measure theoretic approach to the classification of cellular automata

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Abstract

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Each measurable transformation that commutes with a shift on the set of configurations is proved to have an interesting dynamical property, i.e., for almost all initial configurations their orbits have a common structure. We can put those transformations into four classes according to their dynamical properties. The classification of cellular automata is derived from it.

1. Introduction

In [10], Wolfram observed that cellular automata can be put into four classes according to their dynamical behavior. A definition of these four classes based on the evolution on *finite configurations* was given by Culik II and Yu in [3]. In this paper we take another definition based on the fact that *for almost all initial configurations, their orbits have a common dynamical structure.* The author does not know what relations there are between the two ways of definition.

We introduce a class $\mathscr{I}$ consisting of all measurable transformations which commute with at least one of the shifts on the set of configurations with a certain topology and a certain measure on it, and show that each element τ of $\mathscr{I}$ has one of the following four dynamical properties.

(1) For almost every initial configuration c_0, its orbit $\tau^1(c_0), \tau^2(c_0), \ldots$ evolves to a common homogeneous configuration in a bounded time.

(2) For almost every initial configuration c_0, its orbit $\tau^1(c_0), \tau^2(c_0), \ldots$ evolves to an attractor which is cyclic in a sense in a bounded time.

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(3) For almost every initial configuration c_0, its orbit $\tau^1(c_0), \tau^2(c_0), \ldots$ evolves asymptotically to an attractor which is cyclic in a sense.

(4) For almost every initial configuration c_0, its orbit $\tau^1(c_0), \tau^2(c_0), \ldots$ evolves to no simple attractor.

The elements of $\mathscr{I}$ can be put into four classes according to their dynamical properties described above. The classification of cellular automata is derived from that fact.

In Section 2 we define the topology and the measure on the set of configurations, in Section 3 the precise investigation of $\mathscr{I}$ is done, in Section 4 we classify the set of cellular automata and in Section 5 we investigate the second class and the third class in detail.

2. Basic definitions and properties

In this paper, $\mathbb{Z}$ denotes the set of integers, and n and q denote some fixed positive integers. We identify q with the set $\{0, 1, 2, \ldots, q-1\}$ and call it a state set. The symbol $\oplus$ denotes the addition of modulo q. Clarly $\langle q, \oplus \rangle$ is an Abelian group. For any i and j in $\mathbb{Z}$, $i \mid j$ denotes the fact that i is a divisor of j.

Definition 2.1. (1) A map $c : \mathbb{Z}^n \to q$ is called an *(n, q)-configuration*, or merely a *configuration*. $\mathscr{C}$ denotes the set of all configurations.

(2) A configuration which is a constant map is called *homogeneous*. The symbol $\bar{q}_0$ denotes the constant map which takes the value q_0.

We define a topological q-module and a measure on $\mathscr{C}$.

Definition 2.2. (1) $\mathbb{T}$ denotes the product topology of the discrete topology on q, which is induced on $\mathscr{C} = q^{\mathbb{Z}^n}$.

(2) $\mathbb{T}^2$ denotes the product topology of $\mathbb{T}$, which is induced on $\mathscr{C} \times \mathscr{C}$.

Proposition 2.3. *$\langle \mathscr{C}, \mathbb{T} \rangle$ and $\langle \mathscr{C} \times \mathscr{C}, \mathbb{T}^2 \rangle$ are metrizable, complete with such a metric.*

Definition 2.4. The *addition* on $\mathscr{C}$ is defined as follows:

$$(c_0 + c_1)(\alpha) = c_0(\alpha) \oplus c_1(\alpha),$$

for any $c_0, c_1 \in \mathscr{C}$ and $\alpha \in \mathbb{Z}^n$.

Proposition 2.5. *$\langle \mathscr{C}, \mathbb{T}, + \rangle$ is a topological Abelian group.*

Definition 2.6. (1) Let μ_0 be a probability measure on q defined as follows:

$$\mu_0(\{i\}) = \frac{1}{q}, \quad i = 0, \ldots, q-1.$$

μ denotes its *product probability measure* defined on the field of Borel subsets of $\mathscr{C}$.

(2) Ω denotes the field of measurable subsets of $\mathscr{C}$ with respect to the completion of μ.

(3) μ^2 denotes the product probability measure of μ on the field of Borel subsets of $\mathscr{C}\times\mathscr{C}$.

Definition 2.7. Let τ be a transformation on $\mathscr{C}$.

(1) We define τ to be *Borel measurable* if and only if for any Borel measurable subset $\mathscr{B}$ of $\mathscr{C}$ the set $\{c\in\mathscr{C}\colon \tau(c)\in\mathscr{B}\}$ is also Borel measurable.

(2) We define τ to be *measurable* if and only if for any measurable subset $\mathscr{M}$ of $\mathscr{C}$ the set $\{c\in\mathscr{C}\colon \tau(c)\in\mathscr{M}\}$ is also measurable.

Definition 2.8. Let τ_1 and τ_2 be two transformations on $\mathscr{C}$.

(1) We define the *addition* $\tau_1+\tau_2$ and the *complement* $-\tau_1$ pointwisely.

(2) The *composition* $\tau_1\circ\tau_2$ is defined as follows:

$$(\tau_1\circ\tau_2)(c)=\tau_1(\tau_2(c)) \quad \text{for any } c \text{ in } \mathscr{C}.$$

The next proposition is easily derived from the definition.

Proposition 2.9. (1) *Let τ_1 and τ_2 be two Borel measurable transformations on $\mathscr{C}$. The addition $\tau_1+\tau_2$, the complement $-\tau_1$ and the composition $\tau_1\circ\tau_2$ are Borel measurable.*

(2) *Each continuous transformation is Borel measurable.*

Definition 2.10. Let $\beta\in\mathbb{Z}^n$. A *shift* s_β denotes the transformation on $\mathscr{C}$, which is defined as $s_\beta(c)(\alpha)=c(\alpha-\beta)$ for any $c\in\mathscr{C}$, $\alpha\in\mathbb{Z}^n$.

The next lemma is easily proved with the method of ergodic theory [8].

Lemma 2.11. *For each $\beta\in\mathbb{Z}^n-\{\mathbf{0}\}$:*

(1) *The shift s_β is continuous on $\langle\mathscr{C},\mathbb{T}\rangle$.*

(2) *The shift s_β is ergodic on $\langle\Omega,\mathscr{C},\mu\rangle$, i.e.,*

(a) *s_β is a measure preserving transformation on $\langle\Omega,\mathscr{C},\mu\rangle$,*

(b) *if a subset A of $\mathscr{C}$ is measurable and $s_\beta(A)=A$, then $\mu(A)=0$ or 1.*

The next lemma is derived from the continuity of shifts.

Lemma 2.12. *Let c_1, c_2 be two elements of $\mathscr{C}$, $\tau_0,\tau_1,\tau_2,\ldots$ a sequence of Borel measurable transformations on $\mathscr{C}$ and β an element of $\mathbb{Z}^n$. If*

$$\lim_{i\to\infty}\tau_i(c_1)=c_2$$

and

$$s_\beta\tau_i=\tau_i s_\beta$$

for any $i \geq 0$, *then*

$$\lim_{i \to \infty} \tau_i \circ s_\beta(c_1) = s_\beta(c_2).$$

Definition 2.13. Let c be an element of $\mathscr{C}$.

(1) A *subset* $\langle c \rangle$ of $\mathscr{C}$ is defined as follows:

$$\langle c \rangle = \{s_\beta(c) \in \mathscr{C}: \beta \in \mathbb{Z}^n\}.$$

(2) Let c_1 and c_2 be two elements of $\mathscr{C}$. A binary relation $\equiv$ on $\mathscr{C}$ is defined as follows:

$$c_1 \equiv c_2 \quad \text{if and only if} \quad \langle c_1 \rangle = \langle c_2 \rangle.$$

(3) The symbol $\mathscr{C}/s$ denotes the set $\{\langle c \rangle: c \in \mathscr{C}\}$.

Definition 2.14. Let τ be a transformation on $\mathscr{C}$, c_0 an element of $\mathscr{C}$.

(1) For each nonnegative integer m, τ^m is defined inductively as follows:
For each $c \in \mathscr{C}$,

$$\tau^0(c) = c, \tag{1}$$

$$\tau^m(c) = \tau \circ \tau^{m-1}(c) \quad \text{for } m \geq 1. \tag{2}$$

(2) The set $\{\tau^m(c_0): m \geq 1\}$ is called the *orbit* of τ from the initial configuration c_0.

Definition 2.15. Let τ be a Borel measurable transformation on $\mathscr{C}$, T a nonnegative integer, t a positive integer, β an element of $\mathbb{Z}^n$ and q_0 an element of q.

(1) $c \in \mathscr{C}$ is called *(T, q_0)-stable* if and only if the following property holds:

$$\tau^T(c) = \bar{q}_0, \tag{3}$$

$$\tau^{T'+1}(c) = \tau^{T'}(c) \quad \text{for any } T' \geq T. \tag{4}$$

$\mathscr{S}t^{\tau}_{T,q_0}$ denotes the set of all the (T, q_0)-stable configurations. We call $\bar{q}_0$ the limit point from c with the finite τ-iteration.

(2) $c \in \mathscr{C}$ is called *(T, t, β)-cyclic* if and only if the following property holds:

$$\tau^{T'+t}(c) = s_\beta \circ \tau^{T'}(c) \quad \text{for any } T' \geq T.$$

$\mathscr{C}y^{\tau}_{T,t,\beta}$ denotes the set of all the (T, t, β)-cyclic configurations. We call the set

$$\{\langle \tau^{T+i}(c) \rangle: 1 \leq i \leq t\}$$

the limit cycle from c with the finite τ-iteration. Note that the cardinal number of the limit cycle is a divisor of t.

(3) $c \in \mathscr{C}$ is called *(T, t, β)-asymptotically cyclic* if and only if the following property holds:

$$\lim_{T' \to \infty} (\tau^{T'}(c) - \tau^{T'}(c')) = \bar{0}$$

for some $c' \in \mathscr{C}y^{\tau}_{T,t,\beta}$.

$\mathscr{A}c^{\tau}_{T,t,\beta}$ denotes the set of all the (T,t,β)-asymptotically cyclic configurations. We call the set

$$\{\langle \tau^{T+i}(c')\rangle :\ 1 \le i \le t\}$$

the limit cycle from c with the infinite τ-iteration. Note that the cardinal number of the limit cycle is a divisor of t.

We can easily prove the following proposition from the definition above.

Proposition 2.16.

$$\mathscr{S}t^{\tau}_{T,q_0} \subset \mathscr{C}y^{\tau}_{T,1,0}, \tag{5}$$

$$\mathscr{C}y^{\tau}_{T,t,\beta} \subset \mathscr{A}c^{\tau}_{T,t,\beta}. \tag{6}$$

Lemma 2.17. *$\mathscr{S}t^{\tau}_{T,q_0}$, $\mathscr{C}y^{\tau}_{T,t,\beta}$ and $\mathscr{A}c^{\tau}_{T,t,\beta}$ are measurable subsets of $\mathscr{C}$.*

Proof. Clearly $\mathscr{S}t^{\tau}_{T,q_0}$ and $\mathscr{C}y^{\tau}_{T,t,\beta}$ are Borel subsets of $\mathscr{C}$. So they are measurable. $\mathscr{A}c^{\tau}_{T,t,\beta}$ is proved to be a Σ^1_1 subset of $\mathscr{C}$ from the definition. So the Souslin Theorem implies that it is measurable [2]. $\square$

Definition 2.18. (1) A subset $\mathscr{I}$ of the set of Borel measurable transformations on $\mathscr{C}$ is defined as follows. Let τ be a Borel measurable transformation.

$$\tau \in \mathscr{I} \quad \text{if and only if} \quad s_\alpha \tau = \tau s_\alpha \text{ for some } \alpha \in \mathbb{Z}^n - \{0\}.$$

(2) For τ in $\mathscr{I}$, a subset Γ_τ of $\mathbb{Z}^n$ is defined as follows:

$$\Gamma_\tau = \{\alpha \in \mathbb{Z}^n - \{0\} :\ s_\alpha \tau = \tau s_\alpha\}.$$

3. Classification of $\mathscr{I}$

Definition 3.1. The four classes of Borel measurable transformations on $\mathscr{C}$, *CLASS I*, *CLASS II*, *CLASS III* and *CLASS IV* are defined as follows. Let τ be a Borel measurable transformation on $\mathscr{C}$.

(1) $\tau \in$ *CLASS I* if and only if the following property holds. For some $T \ge 0$ and $q_0 \in q$,

$$\mu(\mathscr{S}t^{\tau}_{T,q_0}) = 1.$$

(2) $\tau \in$ *CLASS II* if and only if the following property holds. For any $T \ge 0$ and $q_0 \in q$,

$$\mu(\mathscr{S}t^{\tau}_{T,q_0}) = 0,$$

and for some $T \ge 0$, $t \ge 1$ and $\beta \in \mathbb{Z}^n$,

$$\mu(\mathscr{C}y^{\tau}_{T,t,\beta}) = 1.$$

(3) $\tau \in CLASS\ III$ if and only if the following property holds. For any $T \geq 0$ and $q_0 \in q$,

$$\mu(\mathscr{S}t^{\tau}_{T,q_0}) = 0,$$

for any $T \geq 0$, $t \geq 1$ and $\beta \in \mathbb{Z}^n$,

$$\mu(\mathscr{C}y^{\tau}_{T,t,\beta}) = 0,$$

and for some $T \geq 0$, $t \geq 1$ and $\beta \in \mathbb{Z}^n$,

$$\mu(\mathscr{A}c^{\tau}_{T,t,\beta}) = 1.$$

(4) $\tau \in CLASS\ IV$ if and only if the following property holds. For any $T \geq 0$ and $q_0 \in q$,

$$\mu(\mathscr{S}t^{\tau}_{T,q_0}) = 0,$$

for any $T \geq 0$, $t \geq 1$ and $\beta \in \mathbb{Z}^n$,

$$\mu(\mathscr{C}y^{\tau}_{T,t,\beta}) = 0,$$

and for any $T \geq 0$, $t \geq 1$ and $\beta \in \mathbb{Z}^n$,

$$\mu(\mathscr{A}c^{\tau}_{T,t,\beta}) = 0.$$

Informally speaking, *CLASS I*, for example, is the class of all Borel measurable transformations such that the orbits of almost all initial configurations evolve to a common homogeneous fixed point in a bounded time. The next proposition is easily derived from the definition above.

Proposition 3.2. *Let $i, j = I$, II, III or IV. If $i \neq j$,*

$$CLASS\ i \cap CLASS\ j = \emptyset.$$

Now we can state the classification theorem of $\mathscr{I}$.

Theorem 3.3. *Each element of $\mathscr{I}$ belongs to one and only one of these four classes.*

The next lemma is essential to prove Theorem 3.3.

Lemma 3.4. *Let τ be an arbitrary element of $\mathscr{I}$.*

(1) *For any $T \geq 0$ and $q_0 \in q$,*

$$\mu(\mathscr{S}t^{\tau}_{T,q_0}) = 0 \text{ or } 1.$$

(2) *For any $T \geq 0$, $t \geq 1$ and $\beta \in \mathbb{Z}^n$,*

$$\mu(\mathscr{C}y^{\tau}_{T,t,\beta}) = 0 \text{ or } 1.$$

(3) *For any* $T \geq 0$, $t \geq 1$ *and* $\beta \in \mathbb{Z}^n$,

$$\mu(\mathscr{A}c^{\tau}_{T,t,\beta}) = 0 \text{ or } 1.$$

Proof. We take one fixed element α of Γ_τ.

(1) Clearly the set $(\tau^T)^{-1}(\bar{q}_0)$ is invariant under s_α. So the ergodicity of the shift s_α implies the next equations:

$$\begin{aligned}\mu(\{c \in \mathscr{C}\colon \tau^T(c) = \bar{q}_0\}) &= \mu((\tau^T)^{-1}(\bar{q}_0))\\ &= 0 \text{ or } 1.\end{aligned}$$

On the other hand, for any $T' \geq 0$ the set $(\tau^{T'+1} - s_\beta \circ \tau^{T'})^{-1}(\bar{0})$ is also invariant under s_α. Thus similarly we get the next relation. For any $T' \geq 0$

$$\begin{aligned}\mu(\{c \in \mathscr{C}\colon \tau^{T'+1}(c) = \tau^{T'}(c)\}) &= \mu((\tau^{T'+1} - \tau^{T'})^{-1}(\bar{0}))\\ &= 0 \text{ or } 1.\end{aligned}$$

Thus

$$\begin{aligned}\mu(\mathscr{S}t^{\tau}_{T,q_0}) &= \mu(\{c \in \mathscr{C}\colon \tau^T(c) = \bar{q}_0\} \cap \bigcup_{T' \geq T} \{c \in \mathscr{C}\colon \tau^{T'+1}(c) = \tau^{T'}(c)\})\\ &= 0 \text{ or } 1.\end{aligned}$$

(2) Similarly we get the next relation. For any $T' \geq 0$

$$\begin{aligned}\mu(\{c \in \mathscr{C}\colon \tau^{T'+t}(c) = s_\beta \circ \tau^{T'}(c)\}) &= \mu((\tau^{T'+t} - s_\beta \circ \tau^{T'})^{-1}(\langle \bar{0} \rangle))\\ &= 0 \text{ or } 1.\end{aligned}$$

Hence

$$\begin{aligned}\mu(\mathscr{C}y^{\tau}_{T,t,\beta}) &= \mu(\bigcup_{T' \geq T} \{c \in \mathscr{C}\colon \tau^{T'+t}(c) = s_\beta \circ \tau^{T'}(c)\})\\ &= 0 \text{ or } 1.\end{aligned}$$

(3) Similarly from Lemma 2.12 we can prove that $\mathscr{A}c^{\tau}_{T,t,\beta}$ is invariant under s_α. Therefore

$$\mu(\mathscr{A}c^{\tau}_{T,t,\beta}) = 0 \text{ or } 1. \qquad \square$$

Now we can prove Theorem 3.3.

Proof of Theorem 3.3. Let τ be an element of $\mathscr{I}$.

(1) If $\mu(\mathscr{S}t^{\tau}_{T,q_0}) = 1$ for some $T \geq 0$ and $q_0 \in q$, then $\tau \in$ *CLASS I*.

(2) If $\mu(\mathscr{S}t^{\tau}_{T,q_0}) = 0$ for any $\tilde{T} \geq 0$ and $q_0 \in q$ and $\mu(\mathscr{C}y^{\tau}_{T,t,\beta}) = 1$ for some $T \geq 0$, $t \geq 1$ and $\beta \in \mathbb{Z}^n$, then $\tau \in$ *CLASS II*.

(3) *If* $\mu(\mathscr{S}t^{\tau}_{T,q_0}) = 0$ for any $T \geq 0$ and $q_0 \in q$, $\mu(\mathscr{C}y^{\tau}_{T,t,\beta}) = 0$ for any $T \geq 0$, $t \geq 1$ and $\beta \in \mathbb{Z}^n$ and $\mu(\mathscr{A}c^{\tau}_{T,t,\beta}) = 1$ for some $T \geq 0$, $t \geq 1$ and $\beta \in \mathbb{Z}^n$, then $\tau \in$ *CLASS III*.

(4) If $\mu(\mathscr{S}t^{\tau}_{T,q_0}) = 0$ for any $T \geq 0$ and $q_0 \in q$, $\mu(\mathscr{C}y^{\tau}_{T,t,\beta}) = 0$ for any $T \geq 0$, $t \geq 1$ and $\beta \in \mathbb{Z}^n$ and $\mu(\mathscr{A}c^{\tau}_{T,t,\beta}) = 0$ for any $T \geq 0$, $t \geq 1$ and $\beta \in \mathbb{Z}^n$, then $\tau \in$ *CLASS IV*. $\square$

4. Classification of cellular automata

Definition 4.1. (1) Let r be a positive integer, $\eta_1, \eta_2, \dots, \eta_r$ pairwise distinct elements of $\mathbb{Z}^n$, $\mathcal{N} = \{\eta_1, \eta_2, \dots, \eta_r\}$ and σ a map from q^r into q. The pair $\langle \sigma, \mathcal{N} \rangle$ is called an *n-dimensional cellular automaton with q states*, or merely *CA*. The map σ and set $\mathcal{N}$ are called respectively the *local map* and the *neighborhood indicator* of the CA.

(2) Let $\langle \sigma, \{\eta_1, \eta_2, \dots, \eta_r\} \rangle$ be a CA. The *global transformation* of a CA means the transformation τ on $\mathscr{C}$, which is defined as follows:

$$\tau(c)(\alpha) = \sigma(c(\alpha + \eta_1), \dots, c(\alpha + \eta_r)) \quad \text{for any } c \in \mathscr{C},\ \alpha \in \mathbb{Z}^n.$$

We identify two CA's which have a common global transformation. So, each CA is denoted by its global transformation.

The next theorem is due to Hedlund [5].

Theorem 4.2. *Let τ be a continuous transformation on $\langle \mathscr{C}, \mathbb{T} \rangle$. τ is a CA if and only if $\tau \circ s_\alpha = s_\alpha \circ \tau$ for any $\alpha \in \mathbb{Z}^n$.*

Theorem 4.2 implies that each CA belongs to the class $\mathscr{I}$. Therefore the next theorem is derived from Theorem 3.3.

Theorem 4.3. *Each cellular automaton belongs to CLASS I, CLASS II, CLASS III or CLASS IV.*

The continuity of the cellular automaton τ implies that $\mathscr{S}t^\tau_{T,q_0}$ and $\mathscr{C}y^\tau_{T,t,\beta}$ are closed subsets of $\langle \mathscr{C}, \mathbb{T} \rangle$. The next proposition is easily derived from this fact.

Proposition 4.4. *Let τ be a CA.*

(1) *$\tau \in$ CLASS I if and only if the following property holds. For some $T \geq 0$ and $q_0 \in q$,*

$$\mathscr{S}t^\tau_{T,q_0} = \mathscr{C}.$$

(2) *$\tau \in$ CLASS II if and only if the following property holds. For any $T \geq 0$ and $q_0 \in q$,*

$$\mathscr{S}t^\tau_{T,q_0} \neq \mathscr{C},$$

and for some $T \geq 0$, $t \geq 1$ and $\beta \in \mathbb{Z}^n$,

$$\mathscr{C}y^\tau_{T,t,\beta} = \mathscr{C}.$$

Remark. From the proposition above we get the following relations between Culik–Yu classes I, II and ours [3].

(1) If τ is in *CLASS I* in our sense, then τ is in CLASS I in Culik–Yu classification.

(2) If τ is in *CLASS I* $\cup$ *CLASS II* in our sense, then τ is in CLASS II in Culik–Yu classification.

The author does not know whether the inverse relations hold or not. The relations on the higher classes are also unknown.

A presumptive correspondence between Wolfram's classification and ours is in Table 1.

Table 1
The correspondence between Wolfram's classification and ours

Wolfram's classification	Our classification
CLASS I	*CLASS I*
CLASS II	*CLASS II*
CLASS III	*CLASS IV*
CLASS IV	*CLASS III*

According to the Wolfram's observation [10], his CLASS III and CLASS IV are characterized as follows.

(1) τ is in CLASS III if and only if almost every evolution of τ leads to a chaotic pattern.

(2) τ is in CLASS IV if and only if almost every evolution of τ leads to complex localized structures, sometimes long lived.

Clearly, the character of our *CLASS IV* is nothing but that of his CLASS III described above. Indeed, we can prove that the CA of function code 42 [10] belongs to our *CLASS IV* from the fact that the shift invariant proper subgroup of $\langle \mathscr{C}, + \rangle$ has measure zero.

On the other hand, a close observation of the examples of Wolfram's CLASS IV, function code 20 and 52 [10], show us that a stable or periodic configuration appears after all the "complex localized structures" died in a given space. The "complex localized structure" seems to be a phenomenon based on the way of convergence of point sequences in $\langle \mathscr{C}, \mathbb{T} \rangle$. That is the intuitive reason why we identified his CLASS IV as our *CLASS III*.

It is a difficult question to decide which classes in our sense a given CA belongs to. But from Proposition 4.4, we can say at least that *CLASS I* is recursive enumerable, that is, there is a Turing machine which halts if and only if we input a *CLASS I*-CA to it. Similarly, *CLASS I* $\cup$ *CLASS II* is recursive enumerable.

5. On *CLASS II* and *CLASS III*

Definition 5.1. Let τ be a Borel measurable transformation on $\mathscr{C}$, T a nonnegative integer, t a positive integer, β an element of $\mathbb{Z}^n$ and $c_0, c_1, \ldots, c_{t-1}$ be elements of $\mathscr{C}$ such that $c_i \not\equiv c_j$ when $i \neq j$.

(1) $c \in \mathscr{C}$ is called *$(T, t, \beta; c_0, c_1, \ldots, c_{t-1})$-cyclic* if and only if the following property holds:

$$\tau^{T+i}(c) \text{ is an element of } \langle c_i \rangle, \quad i = 0, 1, \ldots, t-1,$$

$$\tau^{T'+t}(c) = s_\beta \circ \tau^{T'}(c), \quad \text{for any } T' \geq T.$$

$\mathscr{C}y^{\tau}_{T,t,\beta;c_0,c_1,\ldots,c_{t-1}}$ denotes the set of all the $(T, t, \beta; c_0, c_1, \ldots, c_{t-1})$-cyclic configurations.

(2) $c \in \mathscr{C}$ is called *$(T, t, \beta; c_0, c_1, \ldots, c_{t-1})$-asymptotically cyclic* if and only if the following property holds: For some $c' \in \mathscr{C}y^{\tau}_{T,t,\beta;c_0,c_1,\ldots,c_{t-1}}$,

$$\lim_{T' \to \infty} (\tau^{T'}(c) - \tau^{T'}(c')) = \bar{0}.$$

$\mathscr{A}c^{\tau}_{T,t,\beta;c_0,\ldots,c_{t-1}}$ denotes the set of all the $(T, t, \beta; c_0, \ldots, c_{t-1})$-asymptotically cyclic configurations.

We can easily prove the following proposition from the definition above.

Proposition 5.2.

$$\mathscr{C}y^{\tau}_{T,t,\beta} = \bigcup_{t' \mid t;\, t\beta' = t'\beta;\, \langle c_0 \rangle, \ldots, \langle c_{t'-1} \rangle \in \mathscr{C}/s} \mathscr{C}y^{\tau}_{T,t',\beta';c_0,c_1,\ldots,c_{t'-1}}, \tag{7}$$

$$\mathscr{A}c^{\tau}_{T,t,\beta} = \bigcup_{t' \mid t;\, t\beta' = t'\beta;\, \langle c_0 \rangle, \ldots, \langle c_{t'-1} \rangle \in \mathscr{C}/s} \mathscr{A}y^{\tau}_{T,t',\beta';c_0,c_1,\ldots,c_{t'-1}}. \tag{8}$$

We can prove the next lemma in the same way as the proof of Lemma 2.17.

Lemma 5.3. *$\mathscr{C}y^{\tau}_{T,t,\beta;c_0,c_1,\ldots,c_{t-1}}$ and $\mathscr{A}c^{\tau}_{T,t,\beta;c_0,c_1,\ldots,c_{t-1}}$ are measurable subsets of $\mathscr{C}$.*

Lemma 5.4. *Let τ be arbitrary element of $\mathscr{S}$.*

(1) *For any $T \geq 0$, $t \geq 1$ and $c_0, c_1, \ldots, c_{t-1} \in \mathscr{C}$*

$$\mu(\mathscr{C}y^{\tau}_{T,t,\beta;c_0,c_1,\ldots,c_{t-1}}) = 0 \text{ or } 1.$$

(2) *For any $T \geq 0$, $t \geq 1$ and $c_0, c_1, \ldots, c_{t-1} \in \mathscr{C}$*

$$\mu(\mathscr{A}c^{\tau}_{T,t,\beta;c_0,c_1,\ldots,c_{t-1}}) = 0 \text{ or } 1.$$

Proof. We take one fixed element α of Γ_τ.

(1) For $i = 0, 1, 2, \ldots, t-1$, the set $(\tau^{T+i})^{-1}(\langle c_i \rangle)$ is invariant under s_α. So the ergodicity of s_α implies:

$$\mu(\{c\in\mathscr{C}\colon \tau^{T+i}(c)\in\langle c_i\rangle\})=\mu(\tau^{T+i})^{-1}(\langle c_i\rangle))$$
$$=0 \text{ or } 1.$$

On the other hand, for any $T'\geq 0$

$$\mu(\{c\in\mathscr{C}\colon \tau^{T'+t}(c)=s_\beta\circ\tau^{T'}(c)\})=\mu((\tau^{T'+t}-s_\beta\circ\tau^{T'})^{-1}(\langle\bar{0}\rangle))$$
$$=0 \text{ or } 1.$$

Hence

$$\mu(\mathscr{C}y^{\tau}_{T,t,\beta;c_0,c_1,\dots,c_{t-1}})=\mu\Bigg(\bigcup_{i=0,1,\dots,t-1}\{c\in\mathscr{C}\colon \tau^{T+i}(c)\in\langle c_i\rangle\}$$
$$\cap\bigcup_{T'\geq T}\{c\in\mathscr{C}\colon \tau^{T'+t}(c)=s_\beta\circ\tau^{T'}(c)\}\Bigg)$$
$$=0 \text{ or } 1.$$

(2) Similarly we can easily see that the set $\mathscr{A}c^{\tau}_{T,t,\beta;c_0,c_1,\dots,c_{t-1}}$ is invariant under s_α. Therefore

$$\mu(\mathscr{A}c^{\tau}_{T,t,\beta;c_0,c_1,\dots,c_{t-1}})=0 \text{ or } 1. \qquad \square$$

Theorem 5.5.

(1) *For each τ in CLASS II one and only one of the following properties holds:*
   (a) *For almost all initial configurations their orbits evolve to a common limit cycle with the finite iteration.*
   (b) *For almost every initial configuration c_1 and c_2, their limit cycles with the finite iteration are distinct to each other.*

(2) *For each τ in CLASS III one and only one of the following properties holds:*
   (a) *For almost all initial configurations their orbits evolve to a common limit cycle with the infinite iteration.*
   (b) *For almost every initial configuration c_1 and c_2, their limit cycles with the infinite iteration are distinct to each other.*

Proof. (1) Let $\mu(\mathscr{C}y^{\tau}_{T,t,\beta})=1$. If for some $T'\leq T$, $t'\mid t$, $\beta'=(t'/t)\beta$ and $\langle c_0\rangle,\dots,\langle c_{t'-1}\rangle\in\mathscr{C}/s$,

$$\mu(\mathscr{C}y^{\tau}_{T',t',\beta';c_0,c_1,\dots,c_{t-1}})=1.$$

Then clearly (a) holds. So we can assume the following fact. For any $T'\leq T$, $t'\mid t$, $\beta'=(t'/t)\beta$ and $\langle c_0\rangle,\dots,\langle c_{t'-1}\rangle\in\mathscr{C}/s$,

$$\mu(\mathscr{C}y^{\tau}_{T',t',\beta';c_0,c_1,\dots,c_{t-1}})=0.$$

Then for any $T'\leq T$, $t'\mid t$, $\beta'=(t'/t)\beta$ and $\langle c_0\rangle,\dots,\langle c_{t'-1}\rangle\in\mathscr{C}/s$,

$$\mu^2((\mathscr{C}y^{\tau}_{T',t',\beta';c_0,c_1,\dots,c_{t-1}})^2)=0.$$

On the other hand

$$\mu^2((\mathscr{C}y^{\tau}_{T,\iota,\beta})^2) = 1.$$

Therefore (b) holds.

(2) It is proved similarly as above. □

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